Seat No. : _______________

DM-133

December-2017

M.Sc., Sem.-I

402 : Mathematics
(Measure and Integration (New))

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Suppose G is an open subset of [a, b]. Define the length |G|, giving details. True or False ? If G is an open subset of [a, b] and |G| = 0, then G = ϕ. **7**

OR

Suppose F is a closed subset of [a, b]. Define the length |F| and show it is well-defined.

True or false ? If F is a closed subset of [a, b] and |F| = 0, then F = ϕ.

(B) Show that $E \subset [a, b]$ is measurable if and only if given $\varepsilon > 0$ there exist a closed set $F \subset E$ and an open set $G \supset E$ such that $|G| - |F| < \varepsilon$. **7**

OR

Suppose $E \subset [a, b]$. Show that there exists a subset H of E such that H is of type F_σ and $\underline{m}H = \underline{m}E$. (i.e. the inner measures of H and E are equal).

2. (A) Prove that every subset of [a, b] that is of type F_σ is measurable.

Suppose E is the subset of [0, 1] given below. **7**

$$E = \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{8}, \frac{1}{4}\right] \cup \left[\frac{1}{32}, \frac{1}{16}\right] \cup \ldots.. \cup \left[\frac{1}{2^{2k+1}}, \frac{1}{2^{2k}}\right] \cup \ldots.$$

Find mE, the measure of E.

OR

Suppose E_1 and E_2 are subsets of [a, b].

Suppose the symmetric difference of E_1 and E_2 has measure zero. Suppose E_1 is measurable. Show that E_2 is measurable. Show that $mE_1 = mE_2$.

(B) Suppose $f(x) = \frac{1}{x}$, $(0 < x < 1)$ **7**

$f(0) = 6$,

$f(1) = 8$,

Prove that f is measurable on [0, 1]

OR

Suppose f and g are functions on [a, b].

Suppose $f(x) = g(x)$ almost everywhere $(a \le x \le b)$.

Suppose f is measurable.

Show that g is measurable.

3. (A) Suppose f is a bounded measurable function on [a, b]. Show that f is Lebesgue integrable. **7**

OR

Let $f(x) = 3$ $(0 \le x < 1)$,

$f(x) = 5$ $(1 \le x < 2)$,

$f(x) = 4$ $(2 \le x < 3)$,

$f(x) = 3$ $(3 \le x \le 4)$.

(i) Suppose σ is the subdivision {0, 1, 2, 3, 4} of [0, 4]. Calculate $U[f; \sigma]$.

(ii) Let E_k be the inverse image under f of [k, k + 1), k = 3, 4, 5.

Show that $P = \{E_3, E_4, E_5\}$ is a measurable partition of [0, 4].

(iii) Calculate $U[f; P]$ and $L[f; P]$.

(B) Suppose f is a bounded measurable function on [a, b]. Suppose E_1 and E_2 are measurable subsets of [a, b]. Prove that **7**

$$\int_{E_1} f + \int_{E_2} f = \int_{E_1 \cup E_2} f + \int_{E_1 \cap E_2} f$$

OR

Suppose f is a bounded function defined on [a, b]. Suppose f is Lebesgue integrable. Suppose g is a bounded function defined on [a, b] such that

$f(x) = g(x)$ almost everywhere $(a \le x \le b)$.

Show that g is Lebesgue integrable and $\int_a^b g = \int_a^b f$.

4. (A) Suppose $f(x) = \frac{1}{X^P}$ $(0 < x \leq 1)$. **7**

Prove that $f \in \mathcal{L}[0, 1]$ if $p < 1$.

Evaluate the Lebesgue integral $\int_0^1 \frac{1}{X^p}$, for $p < 1$.

OR

Suppose f is a non-negative valued, measurable function on [a, b].

Suppose $f(x) \leq g(x)$ $\quad (a \leq x \leq b)$.

Suppose $g \in \mathcal{L}[a, b]$.

Prove that $f \in \mathcal{L}[a, b]$.

(B) Suppose f is a function defined on [a, b]. **7**

Suppose $f \in \mathcal{L}[a, b]$.

Show that given $\in > 0$, there exists $\delta > 0$

Such that $\left| \int_E f \right| < \in$, whenever E is a measurable subset of [a, b] with $mE < \delta$.

OR

Suppose f is a non-negative valued function in $\mathcal{L}[a, b]$.

Let $E_n = \{x \mid n \leq f(x) < n + 1\}$, $n = 0, 1, 2, \ldots$.

Prove that $\sum_{n=0}^{\infty} n \cdot mE_n < \infty$.

5. Do any **seven**. **14**

(i) Find the outer measure of $E \subset [0, 4]$, where $E = \left[\frac{1}{2}, 1\right] \cup \left[\frac{3}{2}, 2\right] \cup \left[\frac{5}{2}, 3\right]$.

(ii) Find the measure of the set $E \subset [0, 1]$, where $E = \left\{\frac{1}{n} \mid n \in N\right\}$.

(iii) f is defined on [–1, 1] as follows :

$f(x) = -1, \quad (-1 \leq x < 0)$

$f(x) = 1 \ , \quad (0 \leq x \leq 1)$

Show that f is measurable.

(iv) State (without proof) a necessary and sufficient condition for a bounded function to be Riemann integrable on [a, b].

(v) Evaluate $\int_{0}^{\pi} \sin^2 x \, dx$.

(vi) Give an example of a bounded function on [0, 1] which is Lebesgue integrable but not Riemann integrable. (Do not prove).

(vii) State (without proof) the Lebesgue dominated convergence theorem.

(viii) State (without proof) Fatou's lemma.

(x) Suppose $f(x) = \frac{1}{1 + x^2}, \ 0 \le x < \infty$

True or false ? $f \in \mathcal{L}[0, \infty)$. (show work).

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DM-133

December-2017

M.Sc., Sem.-I

402 : Mathematics
(Metric Spaces (Old))

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Define a metric d on a non empty set X. Define an open ball B(X, r) in the metric space (X, d). Define an open set in X. Show that the set $U = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is an open set in the metric space $(\mathbb{R}^2, d_2)$. **7**

OR

Define a metric d on a non-empty set X. Define an open ball B (X, r) in the metric space (X, d). Define an open set in X. Show that the set $F = \{(x, y) \in \mathbb{R}^2 : y \le 0\}$ is a closed set in the metric space $(\mathbb{R}^2, d_2)$.

(B) Answer any **two** : **4**

(i) Consider the metric space $(\mathbb{R}^2, d_2)$. Let $x = (-1, 2)$, $y = (3, 4)$. Find $r > 0$, so that $B(x, r) \cap B(y, r) = \phi$.

(ii) Consider a point b in the metric space (X, d). Show that the set $\{b\}$ is a closed set.

(iii) Is the set of integers $\mathbb{Z}$ a closed subset of $\mathbb{R}$?

(C) Answer **all**. **3**

(i) True or false ? If $d(x, y) = 0$ for points x, y in a metric space (X, d), then $x = y$.

(ii) True or false ? If $d(x, z) = d(y, z)$ for points x, y, z in a metric space (X, d), then $x = y$.

(iii) Find $d_2((1, 2, 3, 4), (0, 1, 2, 3))$ in $(\mathbb{R}^4, d_2)$.

2. (A) Find the limits of the following sequences, if the limit exists. **7**

(i) $\left(\dfrac{n}{n^2 + 1}\right)$ in $\mathbb{R}$.

(ii) $\left(\frac{n}{n^2+1}, \frac{n}{n+1}\right)$ in $(\mathbb{R}^2, d_2)$.

(iii) $\left((-1)^n, 1, \frac{1}{n}\right)$ in $(\mathbb{R}^3, d_2)$.

Justify your answers.

OR

Show that if E is a closed subset of a metric space X, then E contains all its limit points.

Find all the limit points of the subset E of $\mathbb{R}$ where $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(B) Answer any two : **4**

(i) Let $E = \left\{\left(t, \frac{1}{t}\right) : t \in (0, 1)\right\}$. Is E a bounded subset of $\mathbb{R}^2$?

(ii) Is the sequence $\left(\frac{1}{n}\right)$ a Cauchy sequence in $\mathbb{R}$?

(iii) Let $X = \mathbb{R}$ and $A = (0, 1]$. Show that the boundary $\partial A = \{0, 1\}$.

(C) Answer **all**. **3**

(i) Define a dense subset D of a metric space X.

(ii) State (without proof) the Bolzano-Weierstrass Theorem.

(iii) State (without proof) the Weierstrass Approximation Theorem.

3. (A) Suppose (X, d_1) and (Y, d_2) are metric spaces. Consider a function $f : X \to Y$. Define : f is continuous at $x \in X$.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function defined by $f(u, v) = u^2 + v^2$. Show that f is continuous on $\mathbb{R}^2$.

Show that the set of points (u, v) of $\mathbb{R}^2$ such that $u^2 + v^2 \geq 1$ is a closed subset of $\mathbb{R}^2$. **7**

OR

Suppose (X, d_1) and (Y, d_2) are metric spaces. Consider a function $f : X \to Y$.

Define : f is continuous at $x \in X$.

Suppose f is continuous on X. Let F be a closed subset of Y. Show that the set $f^{-1}(F)$ is a closed subset of X.

(B) Answer any **two** : **4**

(i) Suppose $A = \{(u, v) \in \mathbb{R}^2 : u = 2\}$.

Suppose $B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$.

Find d(A, B), the distance between A and B in $\mathbb{R}^2$.

(ii) Show that the interval (0, 1) and the interval (0, 2) of $\mathbb{R}$ are homeomorphic.

(iii) State (without proof) the Tietze extension theorem.

(C) Answer **all**. **3**

(i) Suppose (X, d_1) and (Y, d_2) are metric spaces. Consider a function $f : X \to Y$. Define : f is uniformly continuous on X.

(ii) Give an example of a continuous function from $\mathbb{R}$ to $\mathbb{R}^2$. (do not prove).

(iii) Give an example of a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. (do not prove).

4. (A) Suppose X is a metric space and suppose A is a subset of X. **7**

Define : on open cover of A.

Let K be a subset of X.

Define : K is compact.

State (without proof) the Heine-Borel theorem describing compact subsets of $\mathbb{R}^3$.

Suppose $B = \{(x, y, z) \in \mathbb{R}^3 : |x| + |y| + |z| \leq 1\}$

Is B a compact subset of $\mathbb{R}^3$?

OR

Suppose (X, d) is a metric space. Suppose K is a compact subset of X. Suppose $f : X \to \mathbb{R}$ is a continuous function and $f(x) > 0$ for all $x \in X$. Show that there exist numbers m, M such that $0 < m < f(x) < M$, for all $x \in k$.

(B) Answer any **two** : **4**

(i) Show that an ellipse in $\mathbb{R}^2$ is not homeomorphic to a line.

(ii) Suppose (X, d) is a metric space and K is a compact subset of X. Suppose $A \subset K$ and A is a closed set. Show that A is compact.

(iii) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that f is not bounded.

(C) Answer **all**. **3**

(i) Give an example of an infinite set A which is a compact subset of $\mathbb{R}$. (Do not prove).

(ii) True or false ? Every finite subset of a metric space X is compact. (Do not prove).

(iii) Suppose $A = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 1\}$. Is A a compact subset of $\mathbb{R}^2$? (Do not prove).

5. (A) Define : a connected metric space X. Let $f : X \to \{-1, 1\}$ be a continuous function defined on a metric space X. Show that if X is connected, then f is a constant function.

Suppose A is a connected subset of a metric space X. Show that $\bar{A}$ is connected. ($\bar{A}$ is the closure of A in X). **7**

OR

Suppose (X, d) is a metric space.

Define : a path in X.

Define : X is path-connected.

Suppose X, Y are metric spaces.

Suppose $f : X \to Y$ is a continuous function. Show that the subset f(X) of Y is path-connected, if X is path-connected.

(B) Answer any **two** : **4**

(i) Show that $\mathbb{R}^3$ is path-connected.

(ii) Suppose $A = \{(x, y) : x^2 + y^2 < 1\}$. Is A a path-connected subset of $\mathbb{R}^2$?

(iii) Show that the polynomial $x^3 + 3x + 1$ has a zero in $\mathbb{R}$.

(C) Answer **all**. **3**

(i) Describe (without proof) the connected subsets of $\mathbb{R}$.

(ii) Suppose A is the interval (2, 3) in $\mathbb{R}$. Suppose $B = \{3, 4\}$, a set consisting of two points.

True or false : There is a continuous function from A onto B. (Do not prove).

(iii) Give a subset of $\mathbb{R}^2$ which is not connected. (Do not prove).
